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ON PERIODIC MOTIONS OF A GYROSTAT IN A NEWTONIAN FORCE FIELD*

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The problem of existence of periodic motions of a gyrostat is studied. The gyrostat consists of a rigid body and a rotor, the axis of which is stationary with respect to the rigid body in a central Newtonian force field. In /1,2/, Poincaré's method of small parameter was used to show the existence of periodic motions of a gyrostat with a single fixed point in a central Newtonian force field. It was assumed that the gyrostat differs little from a dynamically symmetric one, and the constant gyrostatic moment was assumed to be sufficiently small. Moreover, it was assumed that the center of gravity of the gyrostat is sufficiently close to the fixed point, and the center of attraction is sufficiently far removed from the gyrostat.

Let us consider a gyrostat /3/ for which the following first integral exists:

$$J(\alpha^{\cdot} + \boldsymbol{\omega}\mathbf{l}_0) = \beta = \text{const}$$

The integral expresses the constancy of the projection of the absolute angular velocity of the rotor on its axis. Here J is the axial moment of inertia of the rotor, $\alpha^{\cdot}$ is the relative angular velocity of the rotor, $\boldsymbol{\omega}$ is the instantaneous angular velocity of the gyrostat and $\mathbf{l}_0$ is the unit vector in the direction of the rotor axis. The generating solution corresponds to free Euler—Poinsot rotational motion, and canonical action—angle variables are chosen as the independent variables.

The Hamiltonian of the problem of motion of a gyrostat about a fixed point in a central Newtonian force field written in terms of the Andoyer variables $l_1, g_1, h_1, L_1, G_1, H_1$ /4/, has the form

$$K = \frac{G_1^2 - L_1^2}{2}\left(\frac{\sin^2 l_1}{A_1} + \frac{\cos^2 l_1}{B_1} + \frac{\sin 2l_1}{D_1}\right) + \frac{L_1^2}{2C_1} + L_1\sqrt{G_1^2 - L_1^2}\left(\frac{\sin l_1}{E_1} + \frac{\cos l_1}{F_1}\right) - \frac{\beta^2}{2J} - U$$

$$U = -P(x_c\gamma_1 + y_c\gamma_2 + z_c\gamma_3) - \frac{3P}{2mR}(A'\gamma_1^2 + B'\gamma_2^2 + C'\gamma_3^2), \quad A_1 = \frac{\Delta}{BC - D^2}, \quad B_1 = \frac{\Delta}{AC - E^2}, \quad C_1 = \frac{\Delta}{AB - F^2}$$

$$D_1 = \frac{\Delta}{C'F}, \quad E_1 = \frac{\Delta}{B'E}, \quad F_1 = \frac{\Delta}{A'D}, \quad \Delta = A'B'C' - J(B'C'l_{01}^2 + A'C'l_{02}^2 + A'B'l_{03}^2)$$

$$A = A' - Jl_{01}^2, \quad B = B' - Jl_{02}^2, \quad C = C' - Jl_{03}^2, \quad D = Jl_{02}l_{03}, \quad E = Jl_{01}l_{03}, \quad F = Jl_{01}l_{02}$$

Here A', B', C' are the principal moments of inertia of the gyrostat about the fixed point; A, B, C are the moments of inertia of the Joukowski-transformed system; l_{01}, l_{02}, l_{03} are the projections of the unit vector $\mathbf{l}_0$ on the moving coordinate axes; L_1 and H_1 are the projections of the kinetic moment $\mathbf{G}_1$ of the gyrostat, relative to the fixed point, on the moving and fixed Z-axis respectively; P and m are the weight and mass of the gyrostat; x_c, y_c and z_c are the coordinates of the center of gravity of the gyrostat in the fixed coordinate system and $\gamma_1, \gamma_2, \gamma_3$ denote the direction cosines of the radius vector of the fixed point R originating at the center of gravity in the moving coordinate system.

We pass to the dimensionless variables $K', L_1', G_1', H_1', x_c', y_c', z_c, t'$ using the formulas

$$K = K'A'\omega_0^2, \quad L_1 = L_1'A'\omega_0, \quad G_1 = G_1'A'\omega_0, \quad H_1 = H_1'A'\omega_0, \quad x_c = x_c'\rho, \quad y_c = y_c'\rho, \quad z_c = z_c'\rho, \quad t' = t\omega_0$$

where ρ is a constant and ω_0 is the initial angular velocity of the gyrostat, assumed to be sufficiently large in module.

Now we assume that the rotor axis is almost parallel to the Z-axis of the moving coordinate system, and introduce a small parameter μ assuming that

$$\frac{P\rho}{A'\omega_0^2} = \mu, \quad \frac{3P}{2mR\omega_0^2} = \varkappa\mu, \quad \frac{\beta^2}{2JA'\omega_0^2} = \lambda\mu, \quad \frac{J}{A'}l_{01} = \nu_1\mu, \quad \frac{J}{A'}l_{02} = \nu_2\mu$$

where $\varkappa, \lambda, \nu_1$ and ν_2 are constants. Omitting the primes in the expressions for the dimensionless variables, we write the Hamiltonian function in the form

$$K = K_0 + \mu K_1, \quad K_0 = \frac{G_1^2 - L_1^2}{2}\left(\sin^2 l_1 + \frac{A'}{B'}\cos^2 l_1\right) + \frac{A'L_1^2}{2(C' - J)}, \quad K_1 = \frac{G_1^2 - L_1^2}{2}(a\sin^2 l_1 + b\cos^2 l_1 + d\sin 2l_1) + \tag{1}$$

$$\frac{L_1^2 c}{2} + L_1\sqrt{G_1^2 - L_1^2}(e\sin l_1 + f\cos l_1) + x_c\gamma_1 + y_c\gamma_2 + z_c\gamma_3 + \varkappa\left(\gamma_1^2 + \frac{B'}{A'}\gamma_2^2 + \frac{C'}{A'}\gamma_3^2\right) - \lambda$$

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Here K_0 is the Hamiltonian defining the generating Euler solution, μK_1 is the perturbing Hamiltonian, and we introduce the following notation:

$$\frac{A'}{A_1}-1=\mu a,\quad A'\left(\frac{1}{B_1}-\frac{1}{B'}\right)=\mu b,\quad \frac{A'}{D_1}=\mu d,\qquad A'\left(\frac{1}{C_1}-\frac{1}{C'-J}\right)=\mu c,\quad \frac{A'}{E_1}=\mu e,\quad \frac{A'}{F'}=\mu f$$

where a, b, c, d, e, f are known constants.

We further assume that the inertia ellipsoid of the gyrostat is almost spherical relative to the fixed point, and pass to the canonical action-angle variables l, g, h, L, G, H. The Hamiltonian of the unperturbed Eulerian motion has the following form in the action-angle variables /5,6/:

$$K_0=\frac{A'\bar{L}^2}{2D'}+\frac{A'G^2}{4}\left(\frac{1}{A'}+\frac{1}{B'}\right) \tag{2}$$

$$\bar{L}=L\left[1+\frac{\varepsilon^2}{16}(b_1-1)(b_1+3)+\frac{\varepsilon^4}{1024}(b_1-1)(3b_1^3-17b_1^2+9b_1+69)+\ldots\right]$$

$$\varepsilon=\frac{D'}{2}\left(\frac{1}{B'}-\frac{1}{A'}\right),\quad \frac{1}{D'}=\frac{1}{C'-J}-\frac{1}{2}\left(\frac{1}{A'}+\frac{1}{B'}\right),\quad b_1=\frac{G^2}{L^2}$$

The Hamiltonian K_0 describes the Eulerian motion of a rigid body with the principal moments of inertia about the fixed point equal to A', B', and $C'-J$. The generating solution described by this motion has the form

$$l_0=n_1^{(0)}t+\omega_1,\quad L_0=a_1,\quad g_0=n_2^{(0)}t+\omega_2,\quad G_0=a_2,\quad h_0=\omega_3,\quad H_0=a_3$$

where ω_i and a_i are arbitrary constants. The frequencies of the angular variables are:

$$n_1^{(0)}=\frac{A'L_0}{D'}\left[1-\frac{\varepsilon^2}{8}(b_0^2+3)-\frac{\varepsilon^4}{512}(15b_0^4-24b_0^3+22b_0^2+51)+\ldots\right]$$

$$n_2^{(0)}=\frac{A'G_0}{2}\left(\frac{1}{A'}+\frac{1}{B'}\right)+\frac{A'G_0\varepsilon^2}{4D'}\left[b_0+1+\frac{\varepsilon^2}{32}(5b_0^3-9b_0^2+11b_0+9)+\ldots\right],\quad b_0=\frac{G_0^2}{L_0^2}$$

When the frequencies $n_1^{(0)}$ and $n_2^{(0)}$ are commensurable, the motion becomes periodic.

To prove the existence of periodic solutions of the system of equations with the Hamiltonian (1), coinciding with the generating solution when $\mu=0$, we must express the perturbing Hamiltonian μK_1 in terms of the action-angle variables. To do this we use the formulas of the unperturbed Eulerian motion /6/ and obtain the following expression, in the form of a series, for the function K_1:

$$K_1=\sum_{s_1,s_2}[a_{s_1,s_2}\sin(s_1l+s_2g)+b_{s_1,s_2}\cos(s_1l+s_2g)]+\sum_{i_1,i_2}b_{i_1,i_2}\cos(i_1l+i_2g)+\sum_{i_1}a_{i_1,0}\sin i_1l-\lambda \tag{3}$$

$$(i_1=0,2,4,\ldots;\ i_2=-2,-1,0,+1,+2;\ s_1=1,3,5,\ldots;\qquad s_2=-1,0,+1)$$

$$a_{s_1,\mp1}=\frac{L\sqrt{G^2-H^2}\,x_c}{2G^2}\theta_{s_1,\mp1},\quad b_{s_1,\mp1}=\frac{L\sqrt{G^2-H^2}\,y_c}{2G^2}\varphi_{s_1,\mp1},\quad a_{s_1,0}=\sqrt{G^2-L^2}\left(\frac{Hx_c}{G^2}\nu_{s_1,0}+Le\varepsilon_{s_1,0}\right) \tag{4}$$

$$b_{s_1,0}=\sqrt{G^2-L^2}\left(\frac{Hy_c}{G^2}\varkappa_{s_1,0}+Lf\mu_{s_1,0}\right)$$

$$b_{i_1,\mp1}=\frac{\sqrt{(G^2-L^2)(G^2-H^2)}}{G^4}\left\{\varkappa\frac{B'-A'}{A'}\left[HL(1-2\delta)d^0_{i_1,\mp1}\mp\frac{H(G\mp L)}{2}d^2_{i_1,\mp1}\right]+G^2z_c\psi_{i_1,\mp1}\right\}$$

$$b_{i_1,0}=\frac{LHz_c}{G^2}\alpha_{i_1,0}+\frac{\varkappa(B'-A')}{4A'G^4}\{G^2(G^2-H^2)(2\delta-1)\delta_{i_1,0}+[(2\delta-1)L^2d^{(0)}_{i_1,0}+(G^2-L^2)d^2_{i_1,0}](3H^2-G^2)\}+$$

$$\frac{G^2-L^2}{2}[(a-b)\varphi_{i_1,0}+b\xi_{i_1,0}]+\frac{L^2c}{2}d^{(0)}_{i_1,0},\qquad a_{i_1,0}=\frac{(G^2-L^2)d}{2}\eta_{i_1,0},\quad \delta=\frac{C'-A'}{B'-A'}$$

Here $\delta_{i_1,0}$ is the Kronecker symbol and the quantities $d^n_{i_1,s_2}$, $\theta_{s_1,\mp1}$, $\nu_{s_1,0}$, $\varepsilon_{s_1,0}$, $\varphi_{s_1,\mp1}$, $\varkappa_{s_1,0}$, $\mu_{s_1,0}$, $\psi_{i_1,\mp1}$, $\alpha_{i_1,0}$, $\varphi_{i_1,0}$, $\xi_{i_1,0}$, $\eta_{i_1,0}$ can be represented by known series in increasing powers of the parameter ε, functions b_1 acting as the coefficients of these series. The coefficients $b_{i_1,\mp2}$ appearing in the expansion of the perturbing Hamiltonian, characterize the inhomogeneity of the field, and coincide with the coefficients $U_{i_1,\mp2}$ of the expansion of the force function U /5/. The perturbing Hamiltonian μK_1 contains, as compared with the Hamiltonian of the problem of motion of a gyrostat, with a constant gyrostatic moment /3/, a new harmonic term $a_{i_1,0}\sin i_1l$.

The equations of motion admit two integrals, the kinetic energy integral and the angular momentum integral, therefore the conditions of existence of periodic Poincaré solutions /7/ reduce to

$$\Delta_1(K_0)\neq 0,\ \frac{\partial [K_1]}{\partial \omega_2}=0,\quad \frac{\partial [K_1]}{\partial a_3}=0,\qquad \Delta_2([K_1])\neq 0,\quad [K_1]=\frac{1}{T}\int_0^T K_1\,dt \tag{5}$$

Here Δ_1 is the Hessian of the unperturbed Hamiltonian relative to a_1 and a_2, and Δ_2 is the Hessian of the mean value of the perturbing Hamiltonian $[K_1]$ relative to a_3 and ω_2. The first condition of (5) always holds, except when $A'=B'=C'-J$, since we have,with the accuracy of up to ε^2,

$$\Delta_1(K_0)=\frac{A'^2}{2D'}\left(\frac{1}{A'}+\frac{1}{B'}\right)+\frac{\varepsilon^2A'^2}{16D'^2}\left[4(3b_0+1)+3D'\left(\frac{1}{A'}+\frac{1}{B'}\right)(b_0^2-1)+\ldots\right]\neq 0$$

In order to inspect the remaining conditions of periodicity, we must obtain the mean value of the function K_1 over a single period. The following cases are possible:

$$1)\ (2N-1)\,n_1^{(0)}=n_2^{(0)},\quad 2)\ 2Nn_1^{(0)}=n_2^{(0)}$$

where N is a positive integer. For the mean value of the function $[K_1]$ we obtain, respectively,

$$1)\ [K_1]=b_{0,0}\left(\frac{L}{G},\frac{H}{G}\right)+a_{2N-1,-1}\sin\eta_1+b_{2N-1,-1}\cos\eta_1+b_{2(2N-1),-2}\cos 2\eta_1-\lambda,\qquad \eta_1=(2N-1)\,\omega_1-\omega_2$$

$$2)\ [K_1]=b_{0,0}\left(\frac{L}{G},\frac{H}{G}\right)+b_{2N,-1}\cos\eta_2+b_{4N,-2}\cos 2\eta_2-\lambda,\qquad \eta_2=2N\omega_1-\omega_2$$

The coefficients $b_{0,0}$, $a_{2N-1,-1}$, $b_{2N-1,1}$, $b_{2(2N-1),-2}$, $b_{2N,-1}$, $b_{4N,-2}$ are defined by the formulas (4) in which the action variables are replaced by their unperturbed values a_1, a_2 and a_3.

Now we can easily write the second condition of (5) explicitly as follows:

$$1)\ a_{2N-1,-1}\cos\eta_1-b_{2N-1,-1}\sin\eta_1-2b_{2(2N-1),-2}\sin 2\eta_1=0,\qquad 2)\ \sin\eta_2\,(b_{2N,-1}+4b_{4N,-2}\cos\eta_2)=0 \tag{6}$$

From (6) we obtain the unperturbed values of the angle variables ω_1 and ω_2. When 1) is commensurable, the third condition of (5) can be written as

$$M_0z_cL_0+H_0\,[M_1+M_2L_0^2+M_3L_0f(H_0)]=0,\qquad f(H_0)=\sqrt{G_0^2-H_0^2} \tag{7}$$

Here M_0, M_1, M_2, M_3 are known constants. We note that when $z_c=0$, then equation (7) is clearly satisfied by the solution $H_0=0$ which, together with the solution $\omega_3=0$, admits a simply geometrical interpretation: the vector of the kinetic moment $\mathbf{G}$ of the gyrostat remains, throughout the whole motion, parallel to the abscissa of the fixed coordinate system. If on the other hand the gyrostat is fixed at the center of mass, then the coefficients $a_{2N-1,-1}$ and $b_{2N-1,-1}$ become zero and we obtain the following unperturbed values for the angle variables: $\omega_1=0$, $\omega_2=0$, $\pi/2$, π, $3\pi/2$, $\omega_3=0$.

When 2) is commensurable we have, in particular, the following unperturbed values of the angle variables: $\omega_1=0$, $\omega_2=0$, π, $\omega_3=0$. If $x_c=y_c=z_c=0$, then the last condition of (5) can be written as

$$(G-L)\,f_1(L)\,f_2(L)\neq 0,\quad f_1(L)=m_1L^2+m_2L+m_3,\qquad f_2(L)=m_4L+m_5 \tag{8}$$

where m_1, m_2, m_3, m_4, m_5 are known constants. Condition (8) breaks down at a finite number of points, namely at $L=G$ and at the points given by the equation $f_1(L)f_2(L)=0$.

Thus we have shown that the problem of motion of a gyrostat in a central Newtonian force field, the study of which was suggested by V. V. Rumiantsev /3/, admits a family of periodic solutions.

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